MATH 218: Elementary Linear Algebra with Applications

Fall 2015-2016, Final, Duration: 120 min.

Exercise 1. Prove or disprove that the following transformations are linear:

(a) **(10 points)** $T_1 : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ defined by $T_1(A) = AA^t$.

(b) **(10 points)** $T_2 : \mathbb{R}_2[X] \to \mathbb{R}$ defined by $T_2(P) = 2P'(0)$.

(c) **(10 points)** $T_3 : \mathbb{R}_1[X] \to \mathbb{R}_1[X]$ with $T_3(1+X) = X$, $T_3(1-X) = 1+X$ and $T_3(2) = 2-X$.

Exercise 2. Let $T : \mathbb{R}_2[X] \to \mathbb{R}_3[X]$ be the linear map defined by

$$T(a + bX + cX^2) = b - c + (a+c)X + (a+b)X^2 + (a+b)X^3.$$

Consider the canonical bases $\mathcal{C}_2 = \{1, X, X^2\}$ and $\mathcal{C}_3 = \{1, X, X^2, X^3\}$.

(a) **(5 points)** Explain briefly and with no computation why T cannot be onto.

(b) **(10 points)** Find a basis of $\ker T$.

(c) **(10 points)** Find a basis of $\mathrm{Im} T$.

(d) **(10 points)** Find the matrix $[T]_{\mathcal{C}_2}^{\mathcal{C}_3}$ of T from $\mathcal{C}_2$ to $\mathcal{C}_3$.

Exercise 3. Let $A = \begin{pmatrix} 2 & 0 & 0 \\ -2 & 2 & 1 \\ 2 & 0 & 1 \end{pmatrix}$.

(a) **(18 points)** Find the eigenvalues and eigenspaces of A.

(b) **(5 points)** Why is A diagonalizable?

(c) **(10 points)** Find P invertible and D diagonal such that $A = PDP^{-1}$ (do not compute P^{-1}).

Exercise 4.

(a) Let $A = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$.

  i. **(5 points)** Show that 1 is an eigenvalue of multiplicity two.

  ii. **(3 points)** Without determining the eigenspace V_1 nor its dimension, explain why A is not diagonalisable (<u>hint:</u> if A was diagonalisable then there would be P, D such that...).

(b) **(5 points)** Let A be a 2×2 matrix. Assume that A has two distinct eigenvalues 1 and -1. Prove that $A^2 = I$.

(c) **(4 points)** Find a 3×3 matrix A with only two eigenvalues 1 and -1 and such that that $A^2 \neq I$.

(d) **(5 points)** Determine one eigenvalue of $A = \begin{pmatrix} -1 & 1 & 1 & 0 & 2 \\ 3 & 2 & -3 & 1 & 0 \\ 2 & 5 & -2 & 0 & 3 \\ -1 & 0 & 1 & 1 & -2 \\ 2 & 2 & -2 & -1 & 1 \end{pmatrix}$.

Exercise 5. Let $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \right\}$.

(a) **(10 points)** Show that $\mathcal{B}$ is a basis of $\mathbb{R}^3$.

(b) **(15 points)** Using the Gram-Schmidt process to transform $\mathcal{B}$ into an orthonormal basis.

Exercise 6. Consider the subspace $W = span\left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$.

(a) **(10 points)** Determine a basis of $W^{\perp}$.

(b) **(5 points)** Let $v = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$. Find $proj_W(v)$.

(c) **(5 points)** Find explicitly $w \in W$ and $w' \in W^{\perp}$ such that $v = w + w'$.

Exercise 7. Assume that $\mathbb{R}^3$ is endowed with the dot product and let $W = span\{w\}$ be a subspace of $\mathbb{R}^3$ of dimension one. Let $proj_W : \mathbb{R}^3 \to \mathbb{R}^3$ be the orthogonal projection on W.

(a) **Part A.** The goal is to show that $proj_W$ is diagonalisable.

i. **(3 points)** Prove that if $u \in W$ then $proj_W(u) = u$ (<u>hint:</u> if $u \in W = span\{w\}$ then u can be written $u = cw$ where c is a scalar).

ii. **(4 points)** Deduce that 1 is an eigenvalue and show that the eigenspace V_1 associated to 1 is equal to W.

iii. **(2 points)** Deduce also from i. that $\mathrm{Im}\, proj_W = W$.

iv. **(4 points)** What is the dimension of $W^{\perp}$?

v. **(3 points)** Prove that $\mathrm{Ker}\, proj_W = W^{\perp}$.

vi. **(3 points)** Use v. to determine another eigenvalue of $proj_W$ and the corresponding eigenspace.

vii. **(3 points)** Why is $proj_W$ diagonalisable?

(b) **Part B.** The goal is to find a basis $\mathcal{B}$ in which $[proj_W]_{\mathcal{B}}$ is diagonal. Assume that $\{v_1, v_2\}$ is an orthogonal basis of $W^{\perp}$.

i. **(4 points)** Prove that $\mathcal{B} = \{w, v_1, v_2\}$ is an orthogonal basis of $\mathbb{R}^3$ (<u>note:</u> recall that we have $W = span\{w\}$).

ii. **(4 points)** Determine the matrix representation $[proj_W]_{\mathcal{B}}$ of $proj_W$ in the basis $\mathcal{B}$

(c) **Part C. (5 points)** Consider the particular case $W = span\left\{ \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right\}$. Find a basis $\mathcal{B}$ such that $[proj_W]_{\mathcal{B}}$ is diagonal (<u>hint:</u> this part is a direct application of Part B; e.g. the first step is to find an orthogonal basis of $W^{\perp}$,...).